Maximum Pagenumber-k Subgraph is NP-Complete

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Abstract

Given a graph G with a total order defined on its vertices, the Maximum Pagenumber-k Subgraph Problem asks for a maximum subgraph G' of G such that G' can be embedded into a k-book when the vertices are placed on the spine according to the specified total order. We show that this problem is NP-complete for $k \geq 2$.

1 Introduction

A *k-book* is a collection of k half-planes, all of which have the same line as their boundary. The half-planes are called the *pages* of the book and the common line is called the *spine*. A *book embedding* is an embedding of a graph into a k-book such that the vertices are placed on the spine, every edge is drawn on a single page, and no two edges cross each other. The *pagenumber* of a graph G is the smallest number of pages for which G has a book embedding.

Computing the pagenumber of a graph is an NP-complete problem [8], and it is even NP-complete to verify if a graph has a certain pagenumber k, for fixed $k \geq 2$. Verifying that a graph has pagenumber 1 however can done in polynomial time: A graph G has pagenumber 1 if and only if it is outerplanar[1] [1], and outerplanarity can be checked in linear time [5].

In certain applications, the order of the vertices along the spines is not arbitrary but specified in the input. In this case we can still check whether a graph has a 1-book embedding that respects $\prec$ in linear time. Assume we have a graph $G = (\{v_1, \ldots, v_m\}, E)$ and spine order $v_1 \prec \cdots \prec v_m$. Extend E with

[1] An undirected graph is *outerplanar* if and only if it has a crossing-free embedding in the plane such that all vertices are on the same face.

the edges $\{\{v_i, v_{i+1}\} \mid 1 \leq i \leq m-1\} \cup \{\{v_m, v_1\}\}$ and note that $G = (V, E)$ can be embedded (in a way that respects $\prec$) into a 1-book if and only if the extended graph is outerplanar. It can also be checked in linear time whether a graph has pagenumber 2 [3].

We are interested in the complexity of the following problem:

(Fixed-Order) Maximum Pagenumber-k Subgraph

Instance: An undirected graph $G = (V, E)$, a total ordering $\prec$ on V, and an integer $m \geq 0$.

Question: Is there a subset $E' \subseteq E$ such that $|E'| \geq m$ and $G' = (V, E')$ can be embedded into a k-book such that the vertices in V are placed on the spine according to the total order $\prec$?

For $k = 1$, this problem can be solved in time $O(|V|^3)$ using dynamic programing [4]. Here we show that, for $k \geq 2$, the problem is NP-complete, and remains so even if we restrict solutions to acyclic subgraphs. That is, the following problem is NP-complete for $k \geq 2$:

Maximum Acyclic Pagenumber-k Subgraph

Instance: A directed graph $G = (V, A)$, a total ordering $\prec$ on V, and an integer $m \geq 0$.

Question: Is there a subset $A' \subseteq A$ such that

1. $|A'| \geq m$,
2. (V, A') is acyclic, and
3. (V, A') can be embedded into a k-book such that the vertices in V are placed on the spine according to the total order $\prec$?

2 Circle graphs

The following is largely based on Unger [9]. A *circle graph* is the intersection graph of a set of chords of a circle. That is, its vertices can be put in one-to-one correspondence with a set of chords in such a way that two vertices are adjacent if and only if the corresponding chords cross each other. An example is shown in Figure 1.

Circle graphs are often represented in a different way. We assume henceforth, without loss of generality, that no two chords have a common endpoint on the circle. To obtain a so-called *overlap model* for a circle graph, we start with a chord drawing and do the following:

1. we break up the circle and straighten it out into a line and
2. we turn the chords into arcs above the line that represents the circle.

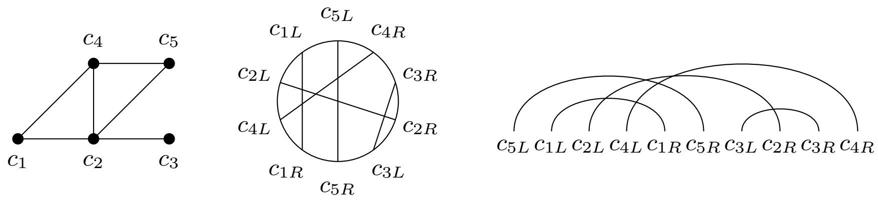

Figure 1: A circle graph (left) together with a corresponding chord drawing (middle) and overlap model (right). The endpoints of a chord c_i are named c_{iL} (left) and c_{iR} (right) based on a counter-clockwise linear ordering of the vertices around the circle (chord drawing) or from left to right (overlap model).

It is easy to see that the standard representation and the overlap representation are equivalent. For simplicity, we will still call the arcs in the overlap representation chords. Given a chord x in an overlap representation, we will denote its left endpoint with x_L and its right endpoint with x_R. We assume without loss of generality that the endpoints are represented by positive integers. Formally, we can now define circle graphs in the overlap representation as follows. An undirected graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ is a circle graph if and only if there exists a set of chords

$$C = \{(c_L, c_R) \mid 1 \leq i \leq n \text{ and } c_L < c_R\}$$

such that

$$\{v_i, v_j\} \in E \quad \text{if and only if} \quad c_i \otimes c_j$$

where the Boolean predicate $\otimes$ denotes the intersection of chords, i.e. $c \otimes d$ if $c_L < d_L < c_R < d_R$ or $d_L < c_L < d_R < c_R$. Given a circle graph $G = (V, E)$ with an overlap representation C, we let $C(v)$ (where $v \in V$) denote the chord corresponding to v.

3 Proof

Given a graph $G = (V, E)$ and a subset $V' \subseteq V$, we let $G|V'$ denote the subgraph of G induced by V', i.e. $G|V$ has vertex set V' and the edge set contains exactly those edges in E' that have both their endpoints in V'.

Our starting point is the following problem.

k-Colourable Induced Subgraph Problem for Circle Graphs (k-CIG)

Instance: A circle graph $G = (V, E)$ and an integer $m \geq 0$.

Question: Is there a subset $V' \subseteq V$ such that $|V'| \geq m$ and the graph $G|V'$ is k-colourable?

Cong and Liu [2] have shown that k-CIG is NP-complete when $k \geq 2$. In the cases when $k = 2$ and $k \geq 4$, this result is based on earlier results by Sarrafzadeh and Lee [6] and Unger [8], respectively.

We now show that MAXIMUM PAGENUMBER-k SUBGRAPH is NP-complete by a reduction from k-CIG.

Proposition 1 MAXIMUM PAGENUMBER-k SUBGRAPH *is NP-complete.*

PROOF We first show that MAXIMUM PAGENUMBER-k SUBGRAPH is in NP. Given an instance $((V, E), \prec, m)$ of this problem, non-deterministically guess a subset $E' \subseteq E$ and a partitioning $E'_1, \ldots, E'_m$ of E'. The instance has a solution if and only if $(V, E'_1), \ldots, (V, E'_k)$ have pagenumber 1 under the spine order $\prec$. This property can be checked in polynomial time as was pointed out earlier.

We next prove NP-hardness via a polynomial-time reduction from k-CIG. Arbitrarily choose a circle graph $G = (V, E)$ and an integer $m \geq 0$. Construct (in polynomial time) an overlap model C of G using the algorithm of Spinrad [7]. This overlap model defines a graph $H = (W, F)$ as follows:

$$W = \{v_L, v_R \mid v \in V \text{ and } C(v) = (v_L, v_R)\}$$
$$F = \{\{v_L, v_R\} \mid v \in V \text{ and } C(v) = (v_L, v_R)\}$$

Finally, let $\prec$ be the natural linear ordering on W. We claim that $(H, \prec, m)$ has a solution if and only if (G, m) has a solution.

Assume that $(H, \prec, m)$ has a solution set of edges X. Assume that $X = X_1 \cup \cdots \cup X_k$ where the edges in X_1 are assigned to page 1, the edges in X_2 to page 2, and so on. Construct a solution set of vertices Y for (G, m) as follows: $Y = \{v \in V \mid \{v_L, v_R\} \in X\}$. Obviously, $|Y| \geq m$. We show that $G|Y$ is k-colourable. Colour the edges in X_i, $1 \leq i \leq k$ with colour i. Each edge corresponds to a vertex in Y; let this vertex inherit its colour. Consider an arbitrary edge (v, w) appearing in $G|Y$. Since $(v, w) \in E$, the chords $C(v) = (v_L, v_R)$ and $C(w) = (w_L, w_R)$ intersect. Thus, the edges $\{v_L, v_R\}$ and $\{w_L, w_R\}$ in F cannot be placed on the same book page which implies that v and w are assigned different colours.

Assume that (G, m) has a solution set of vertices V'. Let $f\colon V' \to \{1, \ldots, k\}$ be a k-colouring of $G|V'$. Construct a solution set of edges T for $(H, \prec, m)$ as follows:

$$T = \{\{v_L, v_R\} \mid v \in V' \text{ and } (v_L, v_R) = C(v)\}$$

Obviously, $|T| \geq m$. We show that (W, T) can be embedded into a k-book with spine order $\prec$. Pick an arbitrary edge $e = \{v_L, v_R\} \in T$ and put e on page $f(v)$. Assume now that edges $e = \{v_L, v_R\}$ and $e' = \{v'_L, v'_R\}$ in T cross each other, i.e. that $(v_L, v_R) \otimes (v'_L, v'_R)$ and e, e' appear on the same page. This implies that $f(v) = f(v')$ and that there is an edge between v and v' in G. Since $v, v' \in V'$, we see that this edge appears in $G|V'$. This contradicts the fact that f is a k-colouring of $G|V'$.

Corollary 1 MAXIMUM ACYCLIC PAGENUMBER-k SUBGRAPH *is NP-hard when* $k \geq 2$.

PROOF Polynomial-time reduction from MAXIMUM PAGENUMBER-k SUBGRAPH. Let $((V, E), \prec, m)$ be an arbitrary instance of MAXIMUM PAGENUMBER-k SUBGRAPH. We first show that MAXIMUM ACYCLIC PAGENUMBER-k SUBGRAPH

is in NP. Non-deterministically guess a subset $A' \subseteq A$ and a partitioning $A'_1, \ldots, A'_m$ of A'. Let E'_i denote the corresponding set of undirected edges, i.e. $E'_i = \{\{v, w\} \mid (v, w) \in A'_i\}$, and note that the instance has a solution if and only if $(V, E'_1), \ldots, (V, E'_k)$ have pagenumber 1 under the spine order $\prec$.

We continue by proving NP-hardness. Assume without loss of generality that $V = \{1, \ldots, m\}$. Construct a directed graph (V, A) as follows: the arc (i, j) is in A if and only if $i < j$ and the edge $\{i, j\}$ is in E. Note that (V, A) (and consequently every subgraph) is acyclic. It is now easy to verify that $((V, A), \prec, m)$ has a solution if and only if $((V, E), \prec, m)$ has a solution.

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